

Functional Quantum Theory of Scattering Processes. III.

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Dynamics of quantum field theory can be formulated by functional equations. To develop a complete functional quantum theory one has to describe the physical information by functional operations only. The most important physical information of elementary particle physics is the S -matrix. In this paper the functional S -matrix is constructed for nonrelativistic spin 1/2 fermions, as in this system a rigorous construction of operator representations is possible. The method of S -matrix derivation used in I and II is improved and the exact perturbation solution for the scattering functionals is given.

In quantum field theory the dynamical behaviour of physical systems can be described by Schwinger functionals of the field operators and corresponding functional equations¹⁻⁷. Using this representation of quantum dynamics it is conclusive to develop a functional quantum theory with appropriate functional spaces, where the information of conventional quantum theory can be obtained by operations in these functional spaces only^{8,9}. One of the most important physical information in quantum field theory is given by the S -matrix. Therefore in functional quantum field theory one has to construct functional representations of the S -matrix. A first step to realize this programme has been made by the introduction of an appropriate scalar product definition for Schwinger functionals^{8,10,11}. Applying this scalar product definition the functional S -matrix definition has been derived for nonrelativistic scattering processes of bosons¹² and for relativistic scattering processes of spin 1/2 fermions¹³ of nonlinear spinor theory¹⁴. As in nonrelativistic quantum field theory a rigorous construction of operator representations is possible, it is instructive to treat this theory in detail in its functional version. Therefore in this paper we discuss the functional S -matrix construction for nonrelativistic scattering processes of spin 1/2 fermions. Additionally corrections to the discussion given in the preceding papers^{12,13} are derived. The improved technic of functional S -matrix construction in this paper leads for nonrelativistic scattering processes instead of formula¹² (3.19) to

$$S_{\mathfrak{M}\mathfrak{N}} = \lim_{\vartheta \rightarrow \infty} \langle \mathfrak{T}_{\mathfrak{M}}^{(-)}(j) | \mathfrak{T}_{\mathfrak{N}}^{(+)}(j) \rangle_{w(\vartheta)}$$

and for relativistic fermion scattering instead of (3.14)¹³ to

$$S_{\mathfrak{M}\mathfrak{N}} = \lim_{\vartheta \rightarrow \infty} \langle \mathfrak{T}_{\mathfrak{M}}^{(-)}(j) | \mathfrak{T}_{\mathfrak{N}}^{(+)}(j) \rangle_{w(\vartheta)} a_1^{-2} \dots a_n^{-2}.$$

This means that in the preceding papers the limes procedure $\vartheta \rightarrow \infty$ has to be added. The definition of the functional symbols and the notation are contained in ^{12, 13}.

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1. Fundamentals

We discuss first the field theoretic description in ordinary i.e. physical Hilbert space. We assume a matter field operator $\psi_\varrho(\mathbf{r}, t)$ satisfying the anticommutation relations

$$\begin{aligned} [\psi_\varrho(\mathbf{r}, t), \psi_{\varrho'}^\dagger(\mathbf{r}', t)]_+ &= \delta_{\varrho\varrho'} \delta(\mathbf{r} - \mathbf{r}'), \\ [\psi_\varrho(\mathbf{r}, t), \psi_{\varrho'}(\mathbf{r}', t)]_+ &= [\psi_\varrho^\dagger(\mathbf{r}, t), \psi_{\varrho'}^\dagger(\mathbf{r}', t)]_+ = 0 \end{aligned} \quad (1.1)$$

where ψ^+ is the Hermitean conjugate of ψ . Using the summation convention for the field system the Hamilton operator

$$H(t) = \int \nabla \psi_\varrho^\dagger(\mathbf{r}, t) \nabla \psi_\varrho(\mathbf{r}, t) d\mathbf{r} + \frac{1}{2} \lambda \int \psi_\varrho^\dagger(\mathbf{r}', t) \psi_\varrho^\dagger(\mathbf{r}, t) V(|\mathbf{r} - \mathbf{r}'|) \psi_\varrho(\mathbf{r}, t) \psi_{\varrho'}(\mathbf{r}', t) d\mathbf{r} d\mathbf{r}' \quad (1.2)$$

is assumed¹². From (1.1) and (1.2) by usual procedures the field equations follow

$$\frac{\partial}{\partial t} \psi_\varrho(\mathbf{r}, t) = \Delta \psi_\varrho(\mathbf{r}, t) - \lambda \int \psi_\varrho^\dagger(\mathbf{r}', t) V(|\mathbf{r} - \mathbf{r}'|) \psi_{\varrho'}(\mathbf{r}', t) \psi_\varrho(\mathbf{r}, t) d\mathbf{r}' \quad (1.3)$$

and the conjugate equations, which result by Hermitean conjugation of (1.3). The representation space of the system can be realized by assuming a true ground state $|0\rangle$ with the property

$$\psi_\varrho(\mathbf{r}, t) |0\rangle = 0. \quad (1.4)$$

The n particle states $|a\rangle$ of the system are then given by

$$|a\rangle := \int \varphi_a \left(\begin{smallmatrix} \mathbf{r}_1 \dots \mathbf{r}_n \\ \varrho_1 \dots \varrho_n \end{smallmatrix} \right) \frac{1}{\sqrt{n!}} \psi_{\varrho_1}^+(\mathbf{r}_1, 0) \dots \psi_{\varrho_n}^+(\mathbf{r}_n, 0) |0\rangle d\mathbf{r}_1 \dots d\mathbf{r}_n \quad (1.5)$$

where the φ_a are the total antisymmetric wave functions of the system. The theory is forminvariant for the Galilei-group and $H(t)$ is the infinitesimal generator of a time translation.

Putting*

$$\psi_\varrho(\mathbf{r}, t) = \psi_{1\varrho}(x), \quad \psi_\varrho^\dagger(\mathbf{r}, t) = \psi_{2\varrho}(x) \quad \text{with} \quad x = \mathbf{r}, t,$$

and combining the two indices into a superindex $\alpha = (\alpha, \varrho)$ the anticommutation relations (1.1) can be reduced to

$$[\psi_\alpha(x), \psi_\beta(x)]_{+|t=t'} = i A_{\alpha\beta} \delta(\mathbf{r} - \mathbf{r}') \quad (1.6)$$

and the field Eqs. (1.3) together with their Hermitean conjugate version go over into

$$\frac{\partial}{\partial t} \psi_\alpha(x) = \Delta D_{\alpha\beta} \psi_\beta(x) + \lambda V_{\alpha\beta\gamma\delta}(x y z u) \psi_\beta(y) \psi_\gamma(z) \psi_\delta(u) \quad (1.7)$$

where the summation convention is now extended to the four variables x with $\int f_\alpha(x) g_\alpha(x) dx = : f_\alpha(x) g_\alpha(x)$. Defining the matrices

$$A := \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \quad C := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad D := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (1.8)$$

the algebraic operators occurring in (1.6) (1.7) are defined by

$$A_{\alpha\beta} := A_{\alpha\varrho\beta\varrho'} = A_{\alpha\beta} \delta_{\varrho\varrho'}, \quad D_{\alpha\beta} := D_{\alpha\varrho\beta\varrho'} = D_{\alpha\beta} \delta_{\varrho\varrho'} \quad (1.9)$$

where all indices have been written explicitly.

The vertex is given in superindices by

$$V_{\alpha\beta\gamma\delta}(x y z u) := D_{\alpha\gamma} C_{\beta\delta} V(|\mathbf{r} - \mathbf{r}'|) \delta(t - t') \delta(x - z) \delta(y - u). \quad (1.10)$$

To define the S -matrix in terms of Heisenberg states, we observe that for any retarded Heisenberg scattering state the relation

$$|a^{(+)}(\mathfrak{N})\rangle \equiv |a^{\text{in}}(\mathfrak{N})\rangle = U(0, -\infty) |\mathfrak{N}\rangle \quad (1.11)$$

* We consider here the operator $\varphi_\alpha = (\psi, \psi^+)$, which in general is not an hermitean operator in contrast to the hermitean operator $\Psi_\alpha = c_{\alpha\beta} \psi_\beta$ in former publications.

is valid with

$$U(0, -\infty) := \lim_{t \rightarrow \infty} e^{iHt} e^{-iH_0 t} =: \Omega^{(+)} \quad (1.12)$$

where $H_0 = \int \nabla \psi_\theta^+(\mathbf{r}, t) \nabla \psi_\theta(\mathbf{r}, t) d\mathbf{r}$ is the Hamilton operator of the free particle Heisenberg state $|\mathfrak{N}\rangle$.

For the advanced Heisenberg scattering states the relation

$$|a^{(-)}(\mathfrak{N})\rangle \equiv |a^{\text{out}}(\mathfrak{N})\rangle = U(0, \infty) |\mathfrak{N}\rangle \quad (1.13)$$

holds with a corresponding definition of $U(0, \infty) =: \Omega^{(-)}$. Then the S -matrix has to be defined by

$$\langle \mathfrak{N} | S | \mathfrak{N}' \rangle := \langle a^{(-)}(\mathfrak{N}) | a^{(+)}(\mathfrak{N}') \rangle = \langle a^{\text{out}}(\mathfrak{N}) | a^{\text{in}}(\mathfrak{N}') \rangle \quad (1.14)$$

which is the Heisenberg representation of the physical definition of the S -matrix resulting from the Schrödinger picture.

To work by the same method in nonrelativistic and relativistic theory we define the Heisenberg field operators⁶

$$\psi_\alpha^{\text{out}}(\mathbf{r}, t) = U^{(\pm)} \psi_\alpha(\mathbf{r}, t) U^{(\pm)-1} \quad (1.15)$$

with

$$U^{(\pm)}(t) = e^{iHt} \Omega^{(\pm)} e^{-iHt} \quad (1.16)$$

which we assume to be unitary, what means that no bound states may occur. From (1.15) we obtain the anticommutation relations

$$[\psi_\alpha^{\text{out}}(\mathbf{r}, t), \psi_{\alpha'}^{\text{out}}(\mathbf{r}', t)]_+ = i A_{\alpha\alpha'} \delta(\mathbf{r} - \mathbf{r}') \quad (1.17)$$

and the equations of motion

$$i \frac{\partial}{\partial t} \psi_\alpha^{\text{out}}(\mathbf{r}, t) = [\psi_\alpha^{\text{out}}(\mathbf{r}, t), H_0^{\text{out}}]_- \quad (1.18)$$

which read explicitly

$$\left(\frac{\partial}{\partial t} \delta_{\alpha\beta} - \Delta D_{\alpha\beta} \right) \psi_\beta^{\text{out}}(\mathbf{r}, t) = 0. \quad (1.19)$$

The connection between the proper field operator and the in (out)-fields can be expressed also by the Yang-Feldman equation

$$\psi_\alpha(x) = \psi_\alpha^{\text{out}}(x) + \lambda G_{\alpha\alpha'}^{(\pm)}(x - x') \cdot V_{\alpha'\beta\gamma\delta}(x' y z u) \psi_\beta(y) \psi_\gamma(z) \psi_\delta(u) \quad (1.20)$$

with the Green-Function $G_{\beta\gamma}^{(\pm)}$ defined by

$$\left(\frac{\partial}{\partial t} \delta_{\alpha\beta} - \Delta D_{\alpha\beta} \right) G_{\beta\gamma}^{(\pm)}(x - x') = \delta_{\alpha\gamma} \delta(x - x') \quad (1.21)$$

and the condition $G_{\beta\gamma}^{(\pm)}(x) = 0$ for $t \geq 0$.

By suitable assumptions about the potential V from (1.20) the limes relation

$$\lim_{t \rightarrow \mp \infty} [\psi_\alpha(\mathbf{r}, t) - \psi_\alpha^{\text{out}}(\mathbf{r}, t)] = 0 \quad (1.22)$$

can be derived in the sense of weak convergence.

2. Asymptotic Free Field Functionals

For the nonrelativistic treatment we assume the interaction picture to be valid. Then the asymptotic particles are well defined. They can be described by functionals. To define them we introduce anti-commuting operators $j_\alpha(x)$ and $\partial_\alpha(x)$ with

$$\begin{aligned} [j_\alpha(x), \partial_{\alpha'}(x')]_+ &= \delta_{\alpha\alpha'} \delta(x - x'), \\ [j_\alpha(x), j_{\alpha'}(x')]_+ &= [\partial_\alpha(x), \partial_{\alpha'}(x')]_+ = 0 \end{aligned} \quad (2.1)$$

and assume a functional ground state $|\varphi_0\rangle$ with

$$\partial_\alpha(x) |\varphi_0\rangle = 0. \quad (2.2)$$

By this assumption it is possible to construct different representation spaces¹⁰. According to^{9,10,15} these spaces have to be representation spaces of the corresponding invariance groups. This condition is satisfied by the construction performed in¹⁰. For the nonrelativistic theory the invariance group is the Galilei-group. The generating Schwinger functionals for the free particle states are then defined by

$$\begin{aligned} \mathfrak{T}_{\mathfrak{N}}^{\text{out}}(j) &:= \langle 0 | T \exp \{ i \psi_\alpha^{\text{out}}(x) j_\alpha(x) \} | a^{\text{out}}(\mathfrak{N}) \rangle \\ &= \sum_{k=1}^{\infty} \frac{i^k}{k!} \langle 0 | T \psi_{\alpha_1}^{\text{out}}(x_1) \dots \psi_{\alpha_k}^{\text{out}}(x_k) | a^{\text{out}}(\mathfrak{N}) \rangle j_{\alpha_1}(x_1) \dots j_{\alpha_k}(x_k). \end{aligned} \quad (2.3)$$

¹⁵ H. STUMPF, K. SCHEERER, and H. G. MÄRTL, Z. Naturforsch. **25 a**, 1556 [1970].

From these the functional states result.

$$|\mathfrak{T}_{\mathfrak{N}}^{\text{out}}(j)\rangle := |\mathfrak{T}_{\mathfrak{N}}^{\text{out}}(j) | \varphi_0\rangle. \quad (2.4)$$

For (2.4) the functional equation

$$\left(\frac{\partial}{i\partial t} \delta_{\alpha\beta} - \Delta D_{\alpha\beta}\right) \mathfrak{C}_{\beta}(x) |\mathfrak{T}_{\mathfrak{N}}^{\text{out}}(j)\rangle = -A_{\alpha\beta} j_{\beta}(x) |\mathfrak{T}_{\mathfrak{N}}^{\text{out}}(j)\rangle \quad (2.5)$$

can be derived by means of (1.17), (1.19)¹⁰. As (2.5) is satisfied for any free field functional the index \mathfrak{N} has been suppressed. It is remarkable that the asymptotic free field functionals are defined with respect to the state vector system of the interacting field. This guarantees the connection with the proper scattering functionals to be defined later. Denoting by $f_{\beta}(x|k)$ a classical solution of the free field Eq. (1.19) and identifying \mathfrak{N} with the complete set of quantum numbers $k_1 \dots k_n$ of n free particles,

the solution of (2.5) is given by¹⁰

$$|\mathfrak{T}_{\mathfrak{N}}^{\text{out}}(j)\rangle \equiv |\mathfrak{T}_{\mathfrak{N}}^{\text{out}}(j, k_1 \dots k_n)\rangle = e^{\frac{1}{2}iFj} \frac{1}{n!} P \sum_{\lambda_1 \dots \lambda_n} (-1)^p f_{\beta_1}(x_1|k_{\lambda_1}) \dots f_{\beta_n}(x_n|k_{\lambda_n}) \times j_{\beta_1}(x_1) \dots j_{\beta_n}(x_n) |\varphi_0\rangle \quad (2.6)$$

where F is the two point function of the corresponding field theory. For the S -matrix construction a scalarproduct is required. According to¹⁰ this can be defined by

$$\begin{aligned} \langle \mathfrak{T}_{\mathfrak{N}}^{\text{out}}(j, k'_1 \dots k'_n) | \mathfrak{T}_{\mathfrak{N}}^{\text{out}}(j, k_1 \dots k_m) \rangle_{w(\vartheta)} : \\ = \langle e^{-\frac{1}{2}j(F+G)j} \mathfrak{T}_{\mathfrak{N}}^{\text{out}}(j, k'_1 \dots k'_n) | e^{-\frac{1}{2}j(F+G)j} \mathfrak{T}_{\mathfrak{N}}^{\text{out}}(j, k_1 \dots k_m) \rangle \end{aligned} \quad (2.7)$$

where the weighting factor G is given by

$$G_{\vartheta}(x, x') = Z_0 g(t - \vartheta) \delta(x - x'). \quad (2.8)$$

The matrix Z_0 has to be skewsymmetric, a suitable choice of g is discussed later.

Denoting the weighted power series functional states by

$$|D_n(x_1 \dots x_n, \beta_1 \dots \beta_n)\rangle := \frac{1}{n!} j_{\beta_1}(x_1) \dots j_{\beta_n}(x_n) e^{-\frac{1}{2}jGj} |\varphi_0\rangle \quad (2.9)$$

and observing that for free fields the operator $e^{-\frac{1}{2}jFj}$ effects normalordering the scalarproduct (2.7) may be written

$$\begin{aligned} \langle \mathfrak{T}_{\mathfrak{N}}^{\text{out}}(j, k'_1 \dots k'_n) | \mathfrak{T}_{\mathfrak{N}}^{\text{out}}(j, k_1 \dots k_m) \rangle_{w(\vartheta)} : \\ = \sum_{l, l'} \langle 0 | N \psi_{\beta'_1}^{\text{out}}(x'_1) \dots \psi_{\beta'_{l'}}^{\text{out}}(x'_{l'}) | a_{k'_1 \dots k'_n}^{\text{out}} \rangle^* \cdot \langle 0 | N \psi_{\beta_1}^{\text{out}}(x_1) \dots \psi_{\beta_l}^{\text{out}}(x_l) | a_{k_1 \dots k_m}^{\text{out}} \rangle \\ \cdot \langle D_{l'}(x'_1 \dots x'_{l'}, \beta'_1 \dots \beta'_{l'}) | D_l(x_1 \dots x_l, \beta_1 \dots \beta_l) \rangle. \end{aligned} \quad (2.10)$$

Straightforward evaluation in analogy to¹⁰ then gives

$$\begin{aligned} \langle \mathfrak{T}_{\mathfrak{N}}^{\text{out}}(j, k'_1 \dots k'_n) | \mathfrak{T}_{\mathfrak{N}}^{\text{out}}(j, k_1 \dots k_m) \rangle_{w(\vartheta)} \\ = P \sum_{\substack{\lambda'_1 \dots \lambda'_n \\ \lambda_1 \dots \lambda_m}} (-1)^{P+P'} f^*(x'_1 | k'_{\lambda'_1}) \dots f^*(x'_n | k'_{\lambda'_n}) f(x_1 | k_{\lambda_1}) \dots f(x_m | k_{\lambda_m}) \\ \times \sum_{\varrho} C(m, n, \varrho) \delta(x'_1 - x_1) a^{-1}(t_1 - \vartheta) \dots \delta(x'_{2\varrho} - x_{2\varrho}) a^{-1}(t_{2\varrho} - \vartheta) \\ \times (-Z_0)^* \delta(x'_{2\varrho+1} - x'_{2\varrho+2}) c(t'_{2\varrho+2} - \vartheta) \dots (-Z_0)^* \delta(x'_{n-1} - x'_n) c(t'_n - \vartheta) \\ \times (-Z_0) \delta(x_{2\varrho+1} - x_{2\varrho+2}) c(t_{2\varrho+2} - \vartheta) \dots (-Z_0) \delta(x_{m-1} - x_m) c(t_m - \vartheta) \end{aligned} \quad (2.11)$$

where the functions $a^{-1}(t - \vartheta)$ and $c(t - \vartheta)$ follow from $g(t - \vartheta)$. By a suitable choice of $g(t - \vartheta)$ one obtains

$$\int f_{\beta}^*(x | k') f_{\beta}(x | k) a^{-1}(t - \vartheta) dx = \delta_{kk'}, \quad (2.12)$$

$$\int f_{\beta}^*(x | k') Z_{0\beta\gamma}^* c(t - \vartheta) f_{\gamma}(x | k) dx = 0. \quad (2.13)$$

Observing $C(n, n, n/2) = (n!)^{-2}$ from (2.9) finally follows

$$\langle \mathfrak{T}_{\mathfrak{N}}^{\text{out}}(j, k'_1 \dots k'_n) | \mathfrak{T}_{\mathfrak{N}}^{\text{out}}(j, k_1 \dots k_m) \rangle = \frac{1}{n!} \delta_{nm} P \sum_{\lambda_1 \dots \lambda_n} (-1)^p \delta_{k'_1 k_{\lambda_1}} \dots \delta_{k'_n k_{\lambda_n}} \quad (2.14)$$

i.e. the orthogonality of the asymptotic free field functionals.

3. Functional S-Matrix Construction

To define the S -matrix for nonrelativistic fermion-fermion scattering in terms of functional scalar-products, we define first the functional scattering states. They are given by

$$|\mathfrak{T}_{\mathfrak{M}}^{(\pm)}(j)\rangle := \langle 0 | T \exp \{i \psi_{\alpha}(x) j_{\alpha}(x)\} | a_{\text{out}}^{\text{in}}(\mathfrak{M}) \rangle | \varphi_0 \rangle. \quad (3.1)$$

Then the following statement holds

Statement: The scattering matrixelement (1.14) for the scattering of spin 1/2 particles of the initial state \mathfrak{N} and the final state \mathfrak{M} is given by

$$S_{\mathfrak{M}\mathfrak{N}} = \lim_{\vartheta \rightarrow \infty} \langle \mathfrak{T}_{\mathfrak{M}}^{(-)}(j) | \mathfrak{T}_{\mathfrak{N}}^{(+)}(j) \rangle_{w(\vartheta)} \quad (3.2)$$

where the scalarproduct has to be performed by the procedure (2.7).

Proof: We consider first the formal normal ordered functional

$$|\Phi_{\mathfrak{N}}^{(\pm)}(j)\rangle := e^{-\frac{1}{2}jFj} |\mathfrak{T}_{\mathfrak{N}}^{(\pm)}(j)\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \langle 0 | N \psi_{\alpha_1}(x_1) \dots \psi_{\alpha_n}(x_n) | a_{\text{out}}^{\text{in}}(\mathfrak{N}) \rangle j_{\alpha_1}(x_1) \dots j_{\alpha_n}(x_n) | \varphi_0 \rangle \quad (3.3)$$

with

$$\begin{aligned} & \langle 0 | N \psi_{\alpha_1}(x_1) \dots \psi_{\alpha_n}(x_n) | a_{\text{out}}^{\text{in}}(\mathfrak{N}) \rangle : \\ &= P \sum_{\lambda_1 \dots \lambda_n} \sum_{r=1}^{[n/2]} \frac{i^r}{2^r r! (n-2r)!} F \binom{x_{\lambda_1} x_{\lambda_2}}{\alpha_{\lambda_1} \alpha_{\lambda_2}} \dots F \binom{x_{\lambda_{2r-1}} x_{\lambda_{2r}}}{\alpha_{\lambda_{2r-1}} \alpha_{\lambda_{2r}}} \langle 0 | T \psi_{\alpha_{2r+1}}(x_{\lambda_{2r+1}}) \dots \psi_{\alpha_n}(x_{\lambda_n}) | a_{\text{out}}^{\text{in}}(\mathfrak{N}) \rangle. \end{aligned} \quad (3.4)$$

Then we have

$$\begin{aligned} \lim_{\vartheta \rightarrow \infty} \langle \mathfrak{T}_{\mathfrak{M}}^{(-)}(j) | \mathfrak{T}_{\mathfrak{N}}^{(+)}(j) \rangle_{w(\vartheta)} &= \lim_{\vartheta \rightarrow \infty} \langle e^{-\frac{1}{2}j(F+G)j} \mathfrak{T}_{\mathfrak{M}}^{(-)}(j) | e^{-\frac{1}{2}j(F+G)j} \mathfrak{T}_{\mathfrak{N}}^{(+)}(j) \rangle \\ &= \lim_{\vartheta \rightarrow \infty} \left(\sum_{n,m} \langle 0 | N \psi_{\beta'_1}(x'_1) \dots \psi_{\beta'_n}(x'_n) | a_{\text{out}}^{\text{out}}(\mathfrak{M}) \rangle^* \right) \\ &\quad \times \langle 0 | N \psi_{\beta_1}(x_1) \dots \psi_{\beta_m}(x_m) | a_{\text{in}}^{\text{in}}(\mathfrak{N}) \rangle \times \left\langle D_n \binom{x'_1 \dots x'_n}{\beta'_1 \dots \beta'_n} \middle| D_m \binom{x_1 \dots x_m}{\beta_1 \dots \beta_m} \right\rangle. \end{aligned} \quad (3.5)$$

The scalarproduct of the weighted power functionals has to be evaluated like for the transition from (2.10) to (2.11)¹⁰. Assuming the functions $a^{-1}(t - \vartheta)$ and $c(t - \vartheta)$ on the compact range $-z_0 \leq t - \vartheta \leq z_0$ the integration for all t -variables in (3.5) is transferred to infinity. But then the asymptotic condition (1.19) can be substituted. This is the technic introduced in ^{12,13}. For the twopoint function we obtain

$$\begin{aligned} \lim_{t_1, t_2 \rightarrow \infty} \langle 0 | T \psi_{\alpha}(x_1) \psi_{\beta}(x_2) | 0 \rangle &= \lim_{t_1, t_2 \rightarrow \infty} \langle 0 | T \psi_{\alpha}^{\text{out}}(x_1) \psi_{\beta}^{\text{out}}(x_2) | 0 \rangle \\ &= \lim_{t_1, t_2 \rightarrow \infty} \langle 0 | T (S_{\text{in}}^{-1} \psi_{\alpha}^{\text{in}}(x_1) S_{\text{in}}) (S_{\text{in}}^{-1} \psi_{\beta}^{\text{in}}(x_2) S_{\text{in}}) | 0 \rangle \\ &= \lim_{t_1, t_2 \rightarrow \infty} \langle 0 | T \psi_{\alpha}^{\text{in}}(x_1) \psi_{\beta}^{\text{in}}(x_2) | 0 \rangle = \lim_{t_1, t_2 \rightarrow \infty} F^{\text{in}} \binom{x_1 x_2}{\alpha \beta} = \lim_{t, t_2 \rightarrow \infty} F^{\text{out}} \binom{x_1 x_2}{\alpha \beta}. \end{aligned} \quad (3.6)$$

Also we obtain with (3.6)

$$\begin{aligned} \lim_{t_1 \rightarrow \infty} \langle 0 | N \psi_{\alpha_1}(x_1) \dots \psi_{\alpha_n}(x_n) | a_{\text{in}}^{\text{in}}(\mathfrak{N}) \rangle &= \lim_{t_1 \rightarrow \infty} \langle 0 | (N \psi_{\alpha_1}^{\text{in}}(x_1) \dots \psi_{\alpha_n}^{\text{in}}(x_n)) S_{\text{in}} | a_{\text{in}}^{\text{in}}(\mathfrak{N}) \rangle \\ &\quad \vdots \\ &\quad \vdots \\ \lim_{t_n \rightarrow \infty} \langle 0 | N \psi_{\alpha_1}(x_1) \dots \psi_{\alpha_n}(x_n) | a_{\text{out}}^{\text{out}}(\mathfrak{M}) \rangle &\quad \vdots \\ &\quad \vdots \\ &\quad \vdots \\ \lim_{t_1 \rightarrow \infty} \langle 0 | N \psi_{\alpha_1}^{\text{out}}(x_1) \dots \psi_{\alpha_n}^{\text{out}}(x_n) | a_{\text{out}}^{\text{out}}(\mathfrak{M}) \rangle &= \lim_{t_1 \rightarrow \infty} \langle 0 | N \psi_{\alpha_1}^{\text{in}}(x_1) \dots \psi_{\alpha_n}^{\text{in}}(x_n) | a_{\text{in}}^{\text{in}}(\mathfrak{N}) \rangle. \end{aligned} \quad (3.7)$$

$$\begin{aligned} &\quad \vdots \\ &\quad \vdots \\ &\quad \vdots \\ &\quad \vdots \\ &\quad \vdots \end{aligned} \quad (3.8)$$

By the use of intermediate states (3.7) can be written

$$\lim_{t_1 \rightarrow \infty} \langle 0 | N \psi_{\alpha_1}(x_1) \dots \psi_{\alpha_n}'(x_n) | a^{\text{in}}(\mathfrak{N}) \rangle = \sum_{\mathfrak{N}} \langle a^{\text{in}}(\mathfrak{N}) | S_{\text{in}} | a^{\text{in}}(\mathfrak{N}) \rangle \lim_{t_1 \rightarrow \infty} \langle 0 | N \psi_{\alpha_1}^{\text{in}}(x_1) \dots \psi_{\alpha_n}^{\text{in}}(x_n) | a^{\text{in}}(\mathfrak{N}) \rangle. \quad (3.9)$$

Observing (2.10) the evaluation of (3.5) by (3.6), (3.7), (3.8), (3.9) finally gives

$$\lim_{\theta \rightarrow \infty} \langle \mathfrak{T}_{\mathfrak{N}}^{(-)}(j) | \mathfrak{T}_{\mathfrak{N}}^{(+)}(j) \rangle_{w(\theta)} = \sum_{\mathfrak{N}} \langle a^{\text{in}}(\mathfrak{N}) | S_{\text{in}} | a^{\text{in}}(\mathfrak{N}) \rangle \lim_{\theta \rightarrow \infty} \langle \mathfrak{T}_{\mathfrak{N}}^{\text{in}}(j) | \mathfrak{T}_{\mathfrak{N}}^{\text{in}}(j) \rangle_{w(\theta)}. \quad (3.10)$$

With

$$\langle a^{\text{in}}(\mathfrak{N}) | S_{\text{in}} | a^{\text{in}}(\mathfrak{N}) \rangle = \langle \mathfrak{N} | S | \mathfrak{N} \rangle \quad (3.11)$$

and (2.14) formula (3.2) follows, q.e.d.

4. Calculational Procedures

An exakt solution for the scattering functionals can be given with (1.20). By the procedure applied in ¹⁰ for any scattering functional (3.1) the functional equation

$$\left[\left(\frac{\partial}{\partial t} \delta_{\alpha\beta} - \Delta D_{\alpha\beta} \right) \partial_{\beta}(x) + A_{\alpha\beta} j_{\beta}(x) \right] | \mathfrak{T}(j) \rangle = \lambda V_{\alpha\beta\gamma\delta}(x y z u) \partial_{\beta}(y) \partial_{\gamma}(z) \partial_{\delta}(u) | \mathfrak{T}(j) \rangle \quad (4.1)$$

can be derived. Substituting the Yang-Feldman Eq. (1.20) into the definition of scattering functionals (3.1) one obtains

$$\begin{aligned} | \mathfrak{T}_{\mathfrak{N}}^{(\pm)}(j) \rangle &= \langle 0 | T \exp \{ i \psi_{\alpha}^{\text{out}}(x) j_{\alpha}(x) + i \lambda G_{\alpha\alpha'}^{(\pm)}(x - x') V_{\alpha'\beta\gamma\delta}(x' y z u) \psi_{\beta}(y) \psi_{\gamma}(z) \psi_{\delta}(u) j_{\alpha}(x) | a^{\text{in}}(\mathfrak{N}) \rangle \\ &= \langle 0 | T \exp \{ i \psi_{\alpha}^{\text{out}}(x) j_{\alpha}(x) \} | a^{\text{out}}(\mathfrak{N}) \rangle + \mathfrak{R}_{\mathfrak{N}}^{(\pm)}(j) \rangle. \end{aligned} \quad (4.2)$$

The second term on the right side can be expanded into a power series in λ containing no zero order terms. Therefore we may write

$$| \mathfrak{R}_{\mathfrak{N}}^{(\pm)}(j) \rangle = \sum_{\varrho=1}^{\infty} \lambda^{\varrho} | \mathfrak{R}_{\mathfrak{N}}^{(\pm)\varrho}(j) \rangle \quad (4.3)$$

and by (2.3) the expression (4.2) can be written

$$| \mathfrak{T}_{\mathfrak{N}}^{(\pm)}(j) \rangle = | \mathfrak{T}_{\mathfrak{N}}^{\text{out}}(j) \rangle + \sum_{\varrho=1}^{\infty} \lambda^{\varrho} | \mathfrak{R}_{\mathfrak{N}}^{(\pm)\varrho}(j) \rangle. \quad (4.4)$$

Substitution of (4.4) into (4.1) and comparison of equal power series coefficients leads to the first order term in λ

$$\left[\frac{\partial}{\partial t} \partial_{\alpha}(x) - \Delta D_{\alpha\beta} \partial_{\beta}(x) + A_{\alpha\beta} j_{\beta}(x) \right] | \mathfrak{R}_{\mathfrak{N}}^{(\pm)1}(j) \rangle = V_{\alpha\beta\gamma\delta}(x y z u) \partial_{\beta}(y) \partial_{\gamma}(z) \partial_{\delta}(u) | \mathfrak{T}_{\mathfrak{N}}^{\text{out}}(j) \rangle \quad (4.5)$$

as $| \mathfrak{T}_{\mathfrak{N}}^{\text{out}}(j) \rangle$ satisfies (2.5). The higher order terms in λ are given by

$$\begin{aligned} \left[\frac{\partial}{\partial t} \partial_{\alpha}(x) - \Delta D_{\alpha\beta} \partial_{\beta}(x) + A_{\alpha\beta} j_{\beta}(x) \right] | \mathfrak{R}_{\mathfrak{N}}^{(\pm)\varrho}(j) \rangle \\ = V_{\alpha\beta\gamma\delta}(x y z u) \partial_{\beta}(y) \partial_{\gamma}(z) \partial_{\delta}(u) | \mathfrak{R}_{\mathfrak{N}}^{(\pm)\varrho-1}(j) \rangle, \quad \varrho = 2, 3, \dots, \infty. \end{aligned} \quad (4.6)$$

To solve this system we make the ansatz

$$| \mathfrak{R}_{\mathfrak{N}}^{(\pm)\varrho}(j) \rangle = e^{\frac{1}{2} i F j} | \chi_{\mathfrak{N}}^{(\pm)\varrho}(j) \rangle \quad (4.7)$$

where F is the twopoint function of the theory. As F satisfies the equation

$$\left[\frac{\partial}{\partial t} \delta_{\alpha\beta} - \Delta D_{\alpha\beta} \right] F_{\beta\gamma}(x - y) = - A_{\alpha\gamma} \delta(x - y) \quad (4.8)$$

this leads to

$$\left[\frac{\partial}{i \partial t} \partial_\alpha(x) - \Delta D_{\alpha\beta} \partial_\beta(x) \right] | \chi_{\mathfrak{N}}^{(\pm)1}(j) \rangle = V_{\alpha\beta\gamma\delta}(x y z u) d_\beta(y) d_\gamma(z) d_\delta(u) | \Phi_{\mathfrak{N}}^{\text{out}}(j) \rangle^{\text{in}} \quad (4.9)$$

where $\Phi_{\mathfrak{N}}^{\text{out}}(j)$ is the normalordered functional and the definition

$$d_\beta(x) := \partial_\beta(x) - F_{\beta\alpha}(x - x') j_\alpha(x') \quad (4.10)$$

is used. For higher order terms we obtain

$$\left[\frac{\partial}{i \partial t} \partial_\alpha(x) - \Delta D_{\alpha\beta} \partial_\beta(x) \right] | \chi_{\mathfrak{N}}^{(\pm)e}(j) \rangle = V_{\alpha\beta\gamma\delta}(x y z u) d_\beta(y) d_\gamma(z) d_\delta(u) | \chi_{\mathfrak{N}}^{(\pm)e-1}(j) \rangle. \quad (4.11)$$

Applying now the Green-Function (1.21) to (4.9), (4.10) and observing that the inhomogeneous term of the scattering functional is already fixed, from (4.9), (4.11) result

$$\begin{aligned} \partial_\alpha(x) | \chi_{\mathfrak{N}}^{(\pm)1}(j) \rangle &= G_{\alpha\alpha'}^{(\pm)}(x - x') V_{\alpha'\beta\gamma\delta}(x' y z u) d_\beta(y) d_\gamma(z) d_\delta(u) | \Phi_{\mathfrak{N}}^{\text{out}}(j) \rangle^{\text{in}}, \\ \partial_\alpha(x) | \chi_{\mathfrak{N}}^{(\pm)e}(j) \rangle &= G_{\alpha\alpha'}^{(\pm)}(x - x') V_{\alpha'\beta\gamma\delta}(x' y z u) d_\beta(y) d_\gamma(z) d_\delta(u) | \chi_{\mathfrak{N}}^{(\pm)e-1}(j) \rangle. \end{aligned} \quad (4.12)$$

Now for the total scattering functional the relation ¹⁵

$$j_\alpha(x) \frac{\partial}{i \partial t} \partial_\alpha(x) | \mathfrak{F}_{\mathfrak{N}}^{(\pm)}(j) \rangle = E_{\mathfrak{N}} | \mathfrak{F}_{\mathfrak{N}}^{(\pm)}(j) \rangle \quad (4.13)$$

holds. As the coupling parameter λ is assumed to be arbitrary at least in a certain range, Eq. (4.13) must be valid even for any term of the expansion (4.4). This can be used to change Eqs. (4.12) into

$$\begin{aligned} | \chi_{\mathfrak{N}}^{(\pm)1}(j) \rangle &= E_{\mathfrak{N}}^{-1} j_\alpha(x) \frac{\partial}{i \partial t} G_{\alpha\alpha'}^{(\pm)}(x - x') V_{\alpha'\beta\gamma\delta}(x' y z u) d_\beta(y) d_\gamma(z) d_\delta(u) | \Phi_{\mathfrak{N}}^{\text{out}}(j) \rangle^{\text{in}}, \\ | \chi_{\mathfrak{N}}^{(\pm)e}(j) \rangle &= E_{\mathfrak{N}}^{-1} j_\alpha(x) \frac{\partial}{i \partial t} G_{\alpha\alpha'}^{(\pm)}(x - x') V_{\alpha'\beta\gamma\delta}(x' y z u) d_\beta(y) d_\gamma(z) d_\delta(u) | \chi_{\mathfrak{N}}^{(\pm)e-1}(j) \rangle. \end{aligned} \quad (4.14)$$

The solution of (4.14) is easily found to be

$$| \chi_{\mathfrak{N}}^{(\pm)e}(j) \rangle = \left[E_{\mathfrak{N}}^{-1} j_\alpha(x) \frac{\partial}{i \partial t} G_{\alpha\alpha'}^{(\pm)}(x - x') V_{\alpha'\beta\gamma\delta}(x' y z u) d_\beta(y) d_\gamma(z) d_\delta(u) \right]^e | \Phi_{\mathfrak{N}}^{\text{out}}(j) \rangle^{\text{in}}. \quad (4.15)$$

By (4.7), (4.4) one therefore obtains for the total scattering functional

$$| \mathfrak{F}_{\mathfrak{N}}^{(\pm)}(j) \rangle = \sum_{e=0}^{\infty} \left(\frac{\lambda}{E_{\mathfrak{N}}} \right)^e \left[j_\alpha(x) \frac{\partial}{i \partial t} G_{\alpha\alpha'}^{(\pm)}(x - x') V_{\alpha'\beta\gamma\delta}(x' y z u) d_\beta(y) d_\gamma(z) d_\delta(u) \right]^e | \mathfrak{F}_{\mathfrak{N}}^{\text{out}}(j) \rangle^{\text{in}} \quad (4.16)$$

which is the perturbation solution in functional formulation.